Journal of Materials and Engineering Structures

Research Paper

A Solution of Plane Stress Problem Subjected to Horizontal Shear Force by Using Polynomial Airy Stress Function

Luu Xuan Le [a, *], Lam Giang To [a], Nghia Trong Nguyen [b], Samir Khatir [c-a], Samir Tiachacht [d], Cuong Thanh Le [b,*]

[a] *Faculty of Civil Engineering, University of Transport and Communications, Vietnam*

[b] *Faculty of Civil Engineering, Ho Chi Minh City Open University, Vietnam*

[c] *Department of Electrical Energy, Metals, Mechanical Constructions and Systems, Faculty of Engineering and Architecture, Ghent University, Belgium*

[d] *Laboratory of Mechanics, Structure and Energetics (LMSE), Mouloud Mammeri University of Tizi-Ouzou, Algeria.*

ARTICLE INFO

Article history :

Received : 17 September 2020

Revised : 21 February 2021

Accepted : 5 May 2021

Keywords:

Plane stress analysis

Airy stress function

Shear stress distribution

Horizontal shear forces

ABSTRACT

Many structural analysis problems in civil engineering and mechanical engineering can be treated as plane stress and plane strain problems introduced in the theory of elasticity. One of the popular analytical methods to tackle plane analysis is to determine Airy stress function. In general, the Airy stress function depends on the analyzed domain and the applied loads; however, the number of problems that can be solved by employing this method is limited because of the formidable challenges of guessing trial function. In many cases, the trial Airy stress functions are selected based on the results of a simple beam model or experimental results. This paper introduces a solution of the plane stress subjected to horizontal shear forces by using a polynomial Airy stress function, in which the trail function is predicted from the results of the elementary beam theory of an equivalent model. The numerical investigation on stress distributions was presented, and it showed that although the internal shear force acting on cross-sections have not appeared, shear stress still appeared, and the shear stress diagram had both negative and positive areas.

1 Introduction

Many problems in elasticity can be reduced from three-dimensional analysis to two-dimensional analysis and generally known as the plane theory of elasticity. Plane stress and strain are typical types involved with this plane analysis. The structures in practice, such as a thin plate with fillet or with hole subjected to loads acting on the plane of the plate and

* *Corresponding author. Tel.: +81 80 9508 0587.*
E-mail address: luusbvl@utc.edu.vn; cuong.lt@ou.edu.vn

e-ISSN: 2170-127X,

RESEARCH REVIEW of Sciences and Technologies

uniformly distributed over the thickness of the plate, are commonly known as plane stress problems [1]. The solution of plane stress analysis in elasticity can be found by satisfying the equation of equilibrium of stress element in conjunction with compatibility equation and boundary conditions. In the case of neglecting body forces, these mentioned equations result in a single formulation called the biharmonic equation which contains the Airy stress function [2]. In this regard, if the Airy stress function is successfully identified satisfying biharmonic equation and boundary conditions, such stress components will be easily obtained through the corresponding derivatives of the Airy stress function. There are two types of Airy stress function commonly used in practice: polynomial function and trigonometric series. Such problems solved by the Airy stress function method have been widely attracted attention in the past. Miki, et al. [3] developed the direct computational implementation of Airy stress functions to solve parametric self-supporting surface problems; Various types of self-supporting surfaces, e.g., symmetric placement of cutting planes, cutting plane placed on the right or left, were computationally investigated. Radice [4] derived a decoupled biharmonic Airy stress function for the solution of stress transfer through the square-end adhesive layer and sandwich structure core. The Airy stress function in [4] is set with two terms related to Closed Form High Order theory (CFHO) and one term related to Decoupled Biharmonic (DB) model, and the coefficients of the Airy stress function are identified on the ground of natural boundary conditions and essential boundary conditions. Without neglecting the body forces, Muti and Dokuz [5] employed the Airy stress function to analyse the stress and strain behaviours of some axially symmetric structures such as thin rotating circular disk and long rotating cylindrical rod, in which such components concerning the body force were presented through potential functions. While the common approach to obtain the coefficients of the Airy stress function is based on the satisfaction of boundary conditions, Cavaco, et al. [6] used the radical displacement data to extract the Airy stress function's coefficients. That experimental technique was particularly used for stress analysis of the soil-pipe interaction of pipelines. In the aspect of Finite Element Method (FEM) for the stress simulation of structures, many types of elements have been successfully developed based on the Airy stress function such as a quadrilateral element with eight nodes of 16 degrees of freedom [7] or triangular element [8]. Many other studies concerning elasticity problems relied on the Airy stress function approach can be found in [9-13] and [14].

Although the Airy stress function method is popular in dealing with the plan stress problems of elasticity, the number of problems that can be solved by employing this method is sometimes limited because of the formidable challenges of guessing trial functions. In many cases, the trial Airy stress functions are selected based on the results of a simple beam model or experimental results.

This paper proposes a solution employing the Airy stress function for a plane stress problem subjected to horizontal shear forces. The unknown components of the polynomial Airy stress function are predicted through the observation of the results of the elementary beam theory of an equivalent beam model. The results of stress analysis by the Airy stress function coincide with those results done by the beam theory. The investigation on stress distributions by a numerical example showed that the shear stress flows appear regardless of no shear force acting on the cross-section.

2 Schematic Diagram and Beam Theory for the Given Problem

2.1 Schematic diagram and problem statement

An analytical scheme for a plane stress problem of the Theory of Elastics is systematically shown in Fig. (1). The cantilever plate with a rectangular cross-section has length l, height h, and thickness t. The upper and lower edges of the plate are subjected to the horizontal shear forces with a constant density of p, which is uniformly distributed over the plate's thickness. The plate is made of the material with Young's modulus E and Poisson's ratio μ. Such fundamental assumptions made for elastic analysis consisting of isotropic body, homogeneous and continuous behaviour of material are valid in this study.

The issue is to determine the functions of stress distributions including normal and shear stresses acting on cross-sectional areas of the plate. The stress spectrum flows and displacements activated along the plate will also be computed and plotted following the developed formulations. As previously mentioned, the estimation of the Airy stress function is paramount important but challenging. To facilitate the prediction of unknown components of the polynomial stress function, the equivalent beam model is firstly analyzed here, then the trial Airy stress function is formulated based on such observations of stress distribution resulted from the beam theory.

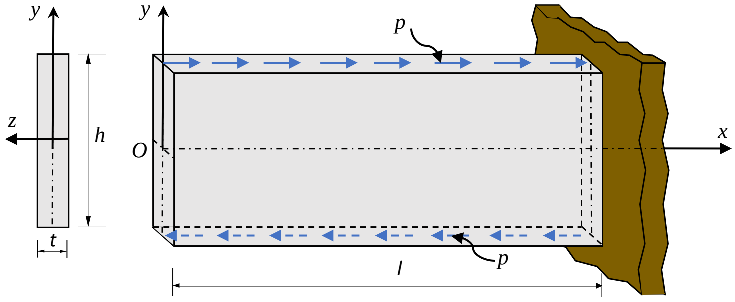

Fig. 1 – Thin plate subjected to horizontal shear forces.

2.2 Beam theory for the given problem

The analysis of an equivalent beam model of the thin plate shown in Fig. 1 yield the internal forces just containing bending moment and zero shear force acting on cross-sectional areas. Because there is no resultant shear force that acts over the cross-section, it could lead to the qualitative conclusion that there are no transverse shear stress distributions on the beam's cross-section as commonly pointed out in Mechanics of Materials [15, 16]. However, this conclusion is incorrect for this given problem, and the below analysis proves that there is an occurrence of shear stresses on the cross-sections of the beam regardless of no internal shear force. The distribution of shear stress contains negative and positive areas so that the integration of the shear stress diagram along with the section' height still results in zero internal shear force. The expressions of normal stress and shear stress based on the equivalent beam theory are following derived.

- Normal stress

$$\sigma_x = -\frac{M_z}{I} y = -\frac{(p \times t \times h \times x)}{\frac{t \times h^3}{12}} y = -\frac{12(p \times x)}{h^2} y \qquad (1)$$

$$\sigma_y = 0 \qquad (2)$$

where y is the perpendicular distance from the neutral axis to the point where the stresses are calculated.

- Shear stress

For a rectangular cross-section with a narrow width, the transverse shear stress is reasonably assumed to be uniformly distributed across the width. Let consider the equilibrium of a beam element dx taken from two adjacent cross-sections as displayed in Fig. 2(a).

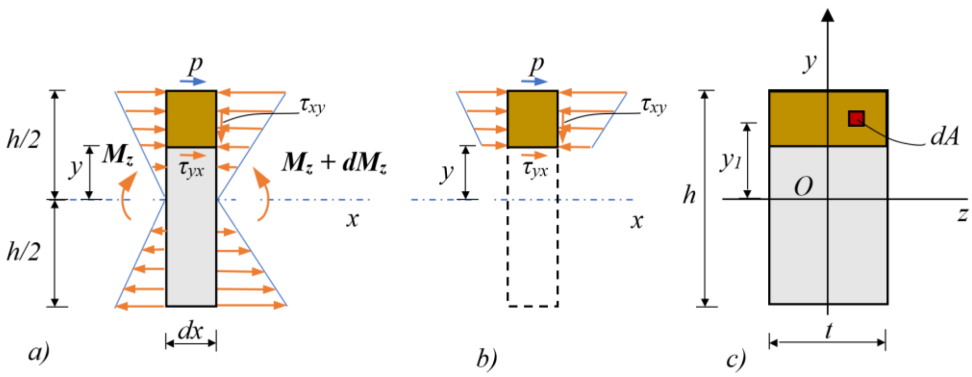

Fig. 2 – Beam element for the derivation of shear stress

The resultant moments acting on two sides of this element are differed by dM_z. the normal stress distributions over the sectional area of each side caused by bending forces are correspondingly shown with triangle distribution diagrams. To evaluate the shear stress, let take an equilibrium of a top portion of the element dx which is sectioned at location y from the neutral axis x as depicted in Fig. (2b). This small portion is subjected to the following horizontal force components: normal stresses at both sides σx, shear stress τ_{xy}, and shear load p.

Taking the force equilibrium of this small portion in the *x*-direction gives

$$\left(\int_y^{h/2}\frac{M_z}{I}\times y_1\times dA\right)+\left(\tau\times t\times dx\right)+\left(p\times t\times dx\right)=\left(\int_y^{h/2}\frac{M_z+dM_z}{I}\times y_1\times dA\right) \quad (3)$$

$$\tau_{yx}=\left(\frac{dM_z}{dx}\int_y^{h/2}y_1 dA\Big/t\times I\right)-p \quad (4)$$

$$\tau_{yx}=\left(\left[\left(p\times t\times h\right)\times\frac{t}{2}\left(\frac{h^2}{4}-y^2\right)\right]\Big/\left[t\times\frac{t\times h^3}{12}\right]\right)-p \quad (5)$$

$$\tau_{yx}=\frac{p}{2}-\frac{6\times p\times y^2}{h^2} \quad (6)$$

Shear stress τ_{xy} acting on the cross-section is finally formed as

$$\tau_{xy}=-\tau_{yx}=-\frac{p}{2}+\frac{6\times p\times y^2}{h^2} \quad (7)$$

Based on the equivalent beam model, the normal stresses σ_x, σ_y and shear stress τ_{xy} are formulated as Eqs. (1), (2) and (7), respectively. It is interesting that although the equivalent beam subjected to horizontal shear force as shown in Fig. 1 triggers zero internal shear force, the transverse shear stress τ_{xy} still exists. The shear stress distributions diagram τ_{xy} plotted using Eq. (7) will contain negative and positive areas so that the integration of the shear stress along the beam' height results in zero internal shear force. The solution of stress problems solved through the approach of the Mechanics of Materials to the equivalent beam will provide hints for the estimation of the trial Airy stress function in the next section.

3 A Solution of Plane Stress Problem Using Polynomial Airy Stress Function

The derivation of fundamental equations of the stress plane problem which are commonly written in the Theory of Elasticity will be introduced herein for the completeness of the work. The plane stress solution for the thin plate subjected to horizontal shear forces as shown in Fig. 1 is proposed by using the Airy stress method.

The solution of plane stress analysis in elasticity, in general, involves three equations: Differential equation of equilibrium; compatibility equation; and boundary conditions. If the body forces are neglected, these formulations reduce to the followings [2]

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}=0;\quad\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{xy}}{\partial x}=0;\quad\nabla^2(\sigma_x+\sigma_y)=0 \quad \text{(8a,b,c)}$$

in which

$$\nabla^2(\sigma_x+\sigma_y)=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(\sigma_x+\sigma_y) \quad (9)$$

The common approach to tackle these equations is to introduce a function so-called the Airy stress function $\Phi(x,y)$. This stress function is related to normal and shear stress components as

$$\sigma_x = \frac{\partial^2\Phi(x,y)}{\partial y^2}; \quad \sigma_y = \frac{\partial^2\Phi(x,y)}{\partial x^2}; \quad \tau_{xy} = -\frac{\partial^2\Phi(x,y)}{\partial x\partial y} \tag{10a,b,c}$$

Substitution of stress components from Eq. (10) into the compatibility equation, Eq. (8c) yields

$$\frac{\partial^4\Phi(x,y)}{\partial x^4} + 2\frac{\partial^4\Phi(x,y)}{\partial x^2\partial y^2} + \frac{\partial^4\Phi(x,y)}{\partial y^4} = 0 \tag{11}$$

Eq. (11) is called the biharmonic equation. Now, the plane stress problem of structural shifts to find the Airy stress function $\Phi(x,y)$ of equation Eq. (11), and satisfy boundary conditions. After having the Airy stress function, the stress components are computed using Eq. 10(a), (b) and (c).

3.1 Derivation of the normal and shear stress equations

In order to derive the formulations of stresses acting on cross-sections of the plate, the Airy stress function method is applied. The polynomial stress function is selected here because of its convenience and simplicity. As mentioned in the preceding section, the formidable challenge of the Airy stress function method is to select a proper trail function. To alleviate that the observation made on the stress distribution results obtained by an equivalent beam model will be displayed here to provide useful information on the trail Airy stress function $\Phi(x,y)$.

Based on the analysis of stress distributions presented in section 2.2, it is clear that the normal stress σ_x is linearly distributed in the x and y direction; the normal stress σ_y is equal to zero at all points; and, the shear stress τ_{xy} is the parabolic function, distributing over cross-sections. Therefore, the polynomial Airy stress function $\Phi(x,y)$ of third-degree as shown below are rationally selected that satisfies those stress observations.

$$\Phi\left(x,y\right) = axy + bxy^3 \tag{12}$$

where a and b are unknown coefficients, which are ascertained by satisfying boundary conditions.

Substitution of Eq. (12) into Eq. (10) gives equations of stresses as

$$\sigma_x = \frac{\partial^2\Phi(x,y)}{\partial y^2} = 6bxy \tag{13}$$

$$\sigma_y = \frac{\partial^2\Phi(x,y)}{\partial x^2} = 0 \tag{14}$$

$$\tau_{xy} = -\frac{\partial^2\Phi(x,y)}{\partial x\partial y} = -a - 3by^2 \tag{15}$$

Boundary conditions

$$\sigma_x\left(x = l, y = h/2\right) = \frac{-6pl}{h} \tag{16}$$

$$\tau_{xy}\left(y = \pm h/2\right) = p \tag{17}$$

By inserting σ_x and τ_{xy} from Eqs. (13) and (15) into Eqs. (16) and (17), respectively, constants of the Airy stress function can be determined, and the Airy stress function is finally formed as

$$\Phi(x,y) = \frac{p}{2}xy - \frac{2p}{h^2}xy^3 \tag{18}$$

Once the Airy stress function is obtained, the normal and shear stress are deduced using Eqs. (13) - (15) as

$$\sigma_x = \frac{\partial^2\Phi(x,y)}{\partial y^2} = -\frac{12p}{h^2}x\,y \tag{19}$$

$$\sigma_y = \frac{\partial^2\Phi(x,y)}{\partial x^2} = 0 \tag{20}$$

$$\tau_{xy} = -\frac{\partial^2\Phi(x,y)}{\partial x\partial y} = -\frac{p}{2} + \frac{6p}{h^2}y^2 \tag{21}$$

It is noted that the stress formulations presented herein only become an exact solution if there are transverse shear forces with the same parabolic distribution as τ_{xy} applied to the free end of the plate, and there are normal forces distributed proportionally to *y*, situated at the fixed support. Without having these two additional force conditions at the free end and support, the proposed solution is not perfectly correct for those cross-sections near both ends of the plate; however, the present solution is still relatively acceptable for those sections far from both ends, according to Saint-Venant's principle. For verification, the normal and shear stresses of the plan stress problem proposed herein coincide completely with the elementary solution of the equivalent beam model.

3.2 Derivation of displacements

The displacements of the plate are formulated here according to the stress expressions developed above. The transverse and horizontal displacements are denoted as $v(x,y)$ and $u(x,y)$, respectively. Applying strain-displacement relations in combination with Hook's law gives

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\sigma_x}{E} = -\frac{12p}{Eh^2}xy \tag{22}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = -\mu\frac{\sigma_x}{E} = \mu\frac{12p}{Eh^2}xy \tag{23}$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{\tau_{xy}}{G} = -\frac{p}{2G} + \frac{6p}{Gh^2}y^2 \tag{24}$$

Integration of Eqs. (22), (23) and (24), together with boundary conditions yield the displacements as

$$v(x,y) = \frac{6\mu p}{Eh^2}xy^2 - \frac{6pl^2}{Eh^2}x + \frac{2p}{Eh^2}x^3 + \frac{4pl^3}{Eh^2} \tag{25}$$

$$u(x,y) = -\frac{6p}{Eh^2}x^2y + \frac{6pl^2}{Eh^2}y - \frac{2\mu p}{Eh^2}y^3 + \frac{2p}{Gh^2}y^3 - \frac{p}{2G}y \tag{26}$$

in which $G = E/\left[2(1+\mu)\right]$ characterizing the shear modulus

Referring to the Appendix for the detailed derivation of displacements.

4 Numerical Investigations

Stress spectrums including normal and shear stress acting on cross-sections of the plate subjected to horizontal shear loadings are presented here, in which normal and shear stress flows are calculated using Eq. (19) and (21), respectively. The plate as shown in Fig. (1) has properties of l = 400 *cm*, h = 100 *cm*, t = 2 *cm*, $E = 2 \times 10^4$ kN/cm^2, and μ = 0.3. The uniformly distributed shear loading applies to the top and bottom faces of the plate in the horizontal direction with a magnitude p = 1 kN/cm^2. As mentioned in the preceding section, the stress distributions are not variable over the thickness of the plate for the plane stress problems, and they are formulations of x and y only. In this investigation, the plate's thickness t is provided for a full description of the plate and implied in the calculation of distributed loadings.

Fig. 3 shows the distributions of normal stress σ_x acting on all cross-sectional areas along the x-axis. The normal stress distribution is proportional to y at each section as shown in Fig. 3(b), and the maximum normal stress of each section occurs at location $y = \pm h/2$.

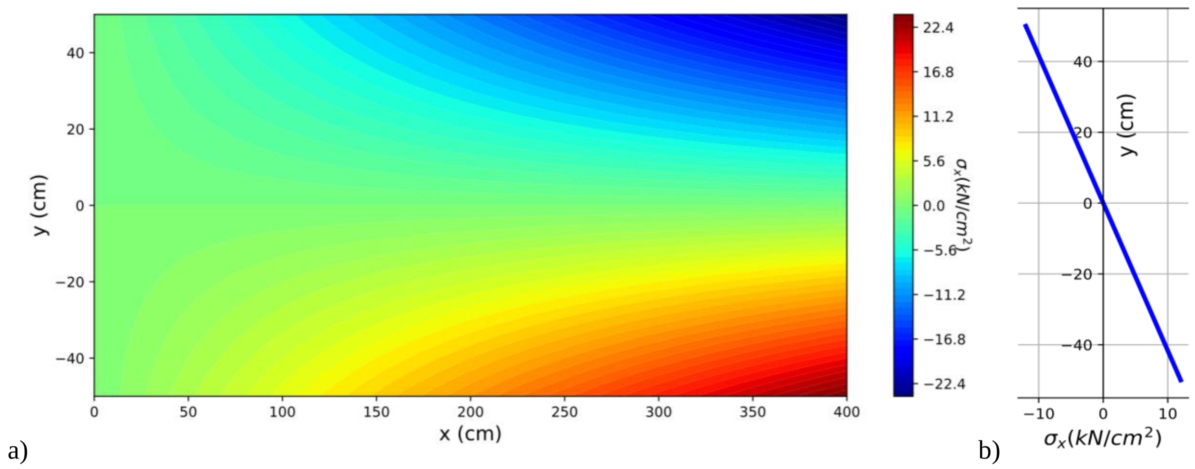

Fig. 3 – Normal stress with Airy stress function: a) Normal stress spectrum; and, b) Normal stress distribution at x = l/2.

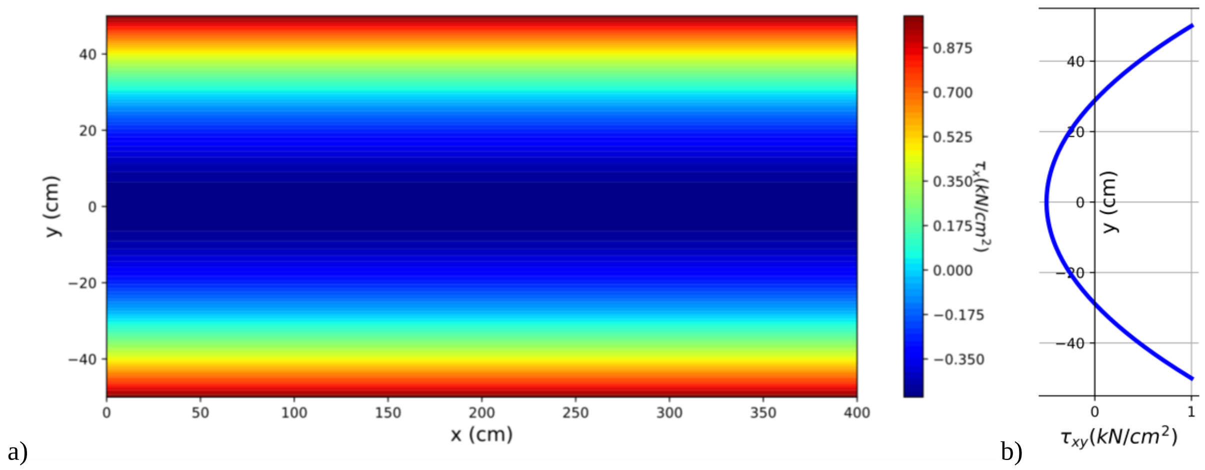

Fig. 4 – Shear stress with Airy stress function: a) Shear stress spectrum; and, b) Shear stress distribution at x = l/2.

Fig. 4 displays the shear stress distributions which have an unchanged parabolic shape along the x-direction. The maximum shear stress takes place at $y = \pm h/2$ with the value as the same as the horizontal shear force value, $\tau_{xy}^{\max} = p$. The stress diagram, Fig. 4(b) pointed out that the shear stress distributes over the sectional area with both positive and negative signs and the integration of τ_{xy} along the plate's height is equal to zero.

Fig. 5 shows transverse displacement $v(x, y)$ of the plate, in which each contour line connects such points on the plate with the same displacement values. For cross-sections located at considerable distances from the fixed ends, the displacements of all points on that section are equal. For the plate's sections near fixed support, the different points along the section's height gain different displacement values. This phenomenon appears as the effect of shear stress distributions on the deflection of the plate. The section at the fixed end is not completely free to distort, and the forces applied to this section are different from those given by Eqs. (19) – (21).

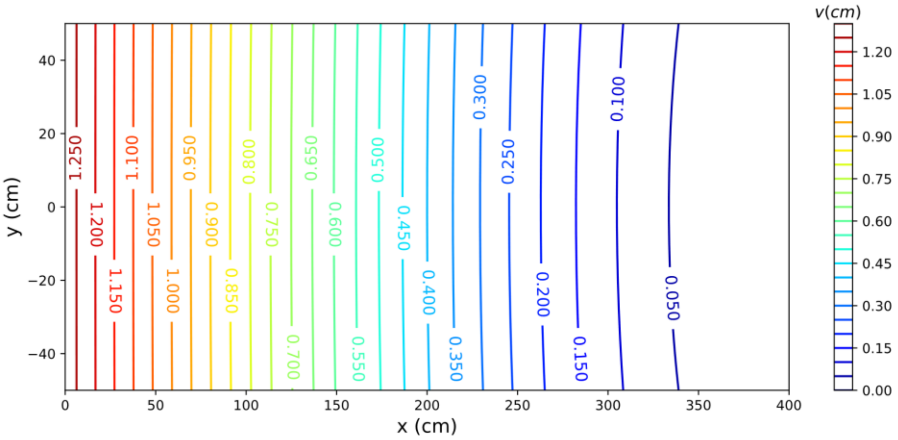

Fig. 5 – Contour lines of transverse displacements v(x,y) of plate

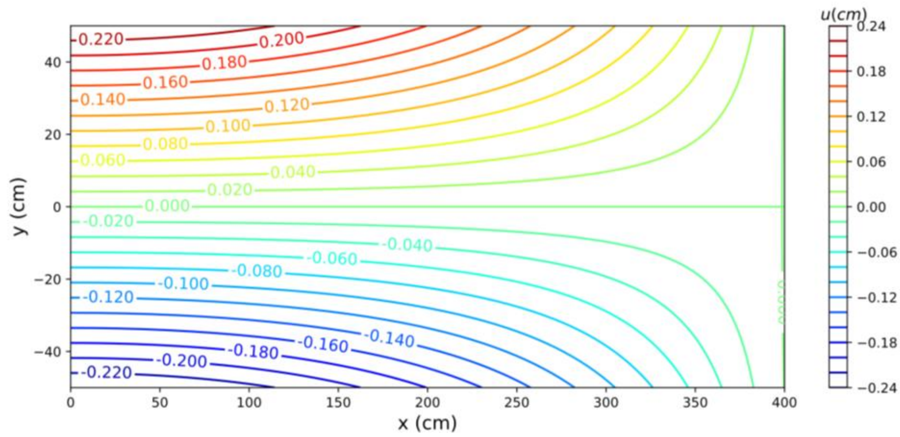

Fig. 6 –Contour lines of horizontal displacements u(x,y) of plate

In a similar manner, Fig. 6 illustrates horizontal displacement *u(x,y)* of the plate with contour line representations. It shows zero displacements at the neutral axis and fixed end section. The maximum horizontal displacements are registered at the free end.

5 Conclusions

The solution of the plane stress problem by using the polynomial Airy stress function is presented in this paper for a thin plate subjected to horizontal shear forces. The structure subjected to horizontal shear loading on the upper and lower edges of the plate poses zero resultant internal shear force. Although there is non-existence of internal shear force, the shear stress still exists on cross-sections. This finding point is useful for stress analysis of similar structures because the Mechanics and Materials commonly point out that if without internal shear force, shear stress will not occur. For the given problem, the difficulty in guessing trial Airy stress function is overcome through the observations of stress analysis resulted from an equivalent beam model of Mechanics of Materials. The shear stress diagram of the thin plate subjected to horizontal shear forces contains negative and positive signs, and the integration of shear stress distribution along with the section's height is equal to zero. The normal stress is linearly distributed over cross-sectional areas. Transverse displacement of all points on a section located at considerable distances from the fixed end are nearly equal, but such sections near the fixed end experience the effect of the fixed constraint leading to unequal values of vertical displacement on that section. There is less effect of boundary restraint on horizontal displacement at ends than that of transverse displacement.

Appendix A. Derivation of Displacements

The derivation of transverse of displacements $v(x,y)$ Eq. (25), and horizontal displacement $u(x,y)$ Eq. (26) are presented here.

Applying Hook's law:

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\sigma_x}{E} = -\frac{12p}{Eh^2}xy;\ \varepsilon_y = \frac{\partial v}{\partial y} = -\mu\frac{\sigma_x}{E} = \mu\frac{12p}{Eh^2}xy;\ \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{\tau_{xy}}{G} = -\frac{p}{2G} + \frac{6p}{Gh^2}y^2 \qquad \text{(27a,b,c)}$$

Integration of Eqs. (27a, b) gives

$$v(x,y) = \frac{6\mu p}{Eh^2}xy^2 + f(x);\ \text{and}\ u(x,y) = -\frac{6p}{Eh^2}x^2y + f(y) \qquad \text{(28a,b)}$$

Substitution of $u(x,y)$ and $v(x,y)$ from Eq. (28a,b) into Eq. (27c) yields

$$\tau_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = [-\frac{6p}{Eh^2}x^2 + \frac{df(x)}{dx}] + [\frac{6\mu\tau_0}{Eh^2}y^2 - \frac{6p}{Gh^2}y^2 + \frac{df(y)}{dy}] = -\frac{p}{2G} \qquad (29)$$

Grouping components of x, and component of y, Eq. (29) leads

$$A(x) + B(y) = -\frac{p}{2G} \qquad (30)$$

here

$$A(x) = [-\frac{6p}{Eh^2}x^2 + \frac{df(x)}{dx}];\ \text{and}\ \mathrm{B}(y) = [\frac{6\mu p}{Eh^2}y^2 - \frac{6p}{Gh^2}y^2 + \frac{df(y)}{dy}] \qquad (31)$$

Because *A(x)* and *B(y)* are functions of *x* and *y* only, *A(x)* and *B(y)* must be constant of *A* and *B*, respectively, to satisfy Eq. 30. Thus,

$$\frac{df(x)}{dx} = A + \frac{6p}{Eh^2}x^2;\ \text{and}\ \frac{df(y)}{dy} = \mathrm{B} - \frac{6\mu p}{Eh^2}y^2 + \frac{6p}{Gh^2}y^2 \qquad (32)$$

Integration of Eq. (32) gives

$$f(x) = Ax + \frac{2p}{Eh^2}x^3 + C;\ \text{and}\ f(y) = \mathrm{B}y - \frac{2\mu p}{Eh^2}y^3 + \frac{2p}{Gh^2}y^3 + D \qquad \text{(33a,b)}$$

Substitution of *f(x)* and *f(y)* into Eq. (28) results in

$$v(x,y) = \frac{6\mu p}{Eh^2}xy^2 + Ax + \frac{2p}{Eh^2}x^3 + C \qquad (34)$$

$$u(x,y) = -\frac{6p}{Eh^2}x^2y + \mathrm{B}y - \frac{2\mu p}{Eh^2}y^3 + \frac{2p}{Gh^2}y^3 + D \qquad (35)$$

The four constants A, B, C and D appeared in Eqs. (34) and (35) are determined using three conditions of constraint together with Eq. (30). Let consider the point *K* at the centroid of a cross-section at the fixed end. Then $u(x,y) = v(x,y) = 0$ at $x = l$ and $y = 0$. The third condition is that the element of the axis of the plate is fixed, thus

$$\left(\frac{\partial v}{\partial x}\right)_{x=l,y=0} = 0 \qquad (36)$$

Solving the four above mentioned conditions gives the constants as

$$A=-\frac{6pl^2}{Eh^2};\ B=-\frac{p}{2G}+\frac{6pl^2}{Eh^2};\ C=\frac{4pl^3}{Eh^2};\text{ and }D=0 \tag{37}$$

By inserting A, B, C and D into Eq. (34) and (35), the transverse displacement $v(x,y)$ and horizontal displacement $u(x,y)$ are correspondingly derived.

REFERENCES

[1]- P.L. Gould, Introduction to linear elasticity. New York: Springer-Verlag. 2nd edition, 1999.

[2]- S. Timoshenko, J.N. Goodier, Theory of Elasticity. New York: McGraw - Hil1. Fourth Edition, 1970.

[3]- M. Miki, T. Igarashi, P. Block, Parametric self-supporting surfaces via direct computation of airy stress functions. ACM Trans. Graph. 34(4) (2015) Article 89. doi:10.1145/2766888.

[4]- J. Radice, On the decoupled biharmonic airy stress function for the square-end adhesive layer and sandwich structure core. J. Sandwich Struct. Mater. 23(1) (2021) 23-46. doi:10.1177/1099636218818624.

[5]- S. Muti, M.S. Dokuz, Two-dimensional Beltrami–Michell equations for a mixture of two linear elastic solids and some applications using the Airy stress function. Int. J. Solids Struct. 59(2015) 140-146. doi:10.1016/j.ijsolstr.2015.01.022.

[6]- M.A.M. Cavaco, M.R. Viotti, G.A. Albertazzi, Jr., Pipeline Stress Analysis Through Stress Function Fittings. Lat. Am. J. Solid Struct. 15(9) (2018) e92:1-11. doi:10.1590/1679-78254812.

[7]- A. Madeo, R. Casciaro, G. Zagari, R. Zinno, G. Zucco, A mixed isostatic 16 dof quadrilateral membrane element with drilling rotations, based on Airy stresses. Finite Elem. Anal. Des. 89 (2014) 52-66. doi:10.1016/j.finel.2014.05.013.

[8]- M. Rezaiee-Pajand, A. Karimipour, Three stress-based triangular elements. Eng. Comput., 36(4) (2020) 1325-1345. doi:10.1007/s00366-019-00765-6.

[9]- C.C. Ike, Elzaki transform method for finding solutions to two-dimensional elasticity problems in polar coordinates formulated using Airy stress functions. J. Comput. Appl. Mech. 51(2) (2020) 302-310. doi:10.22059/jcamech.2020.296012.472.

[10]- Y. Shang, S. Cen, M.-J. Zhou, 8-node unsymmetric distortion-immune element based on Airy stress solutions for plane orthotropic problems. AcMec, 229(12) (2018) 5031-5049. doi:10.1007/s00707-018-2291-3.

[11]- J. Gahleitner, J. Schoeftner, An anisotropic beam theory based on the extension of Boley's method. Compos. Struct., 243 (2020) 112149. doi:10.1016/j.compstruct.2020.112149.

[12]- P. Chu, X.F. Li, J.X. Wu, K.Y. Lee, Two-dimensional elasticity solution of elastic strips and beams made of functionally graded materials under tension and bending. AcMec, 226(7) (2015) 2235-2253. doi:10.1007/s00707-014-1294-y.

[13]- Y. Lian, X. He, S. Shi, X. Li, Z. Yang, J. Sun, A Multi-Parameter Perturbation Solution for Functionally Graded Piezoelectric Cantilever Beams under Combined Loads. Materials 11(7) (2018) 1222. doi:10.3390/ma11071222.

[14]- H. Nazif, L.S. Lee, Optimised crossover genetic algorithm for capacitated vehicle routing problem. Appl. Math. Model. 36(5) (2012) 2110-2117. doi:10.1016/j.apm.2011.08.010.

[15]- R.C. Hibbeler, Mechanics of Materials. Pearson Prentice Hall. Eighth Edition, 2011.

[16]- J.M. Gere, S.P. Timoshenko, Mechanics of materials. Chapman and Hall. Third SI Edition, 1991.